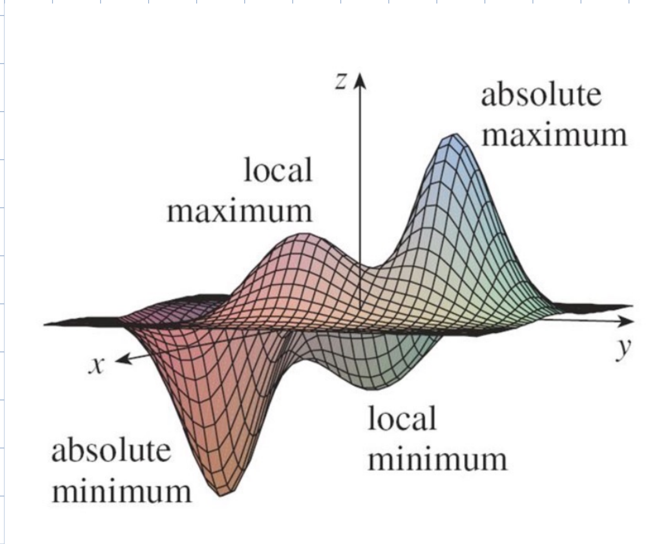
Last time:



Thm: If f has a local maximum or minimum at (a,b) then
$$f_x(a,b)=0 \text{ and } f_y(a,b)=0.$$

Rmk: i.e. $\nabla f(a,b)=0$

Def: If $\nabla f(a,b)=0$, we call (a,b) a critical point of f.

Second derivatives test

Suppose (a,b) is a critical point of f (i.e. $f_x(a,b)=0$ and $f_y(a,b)=0$).

Set $D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - (f_{xy}(a,b))^2$

(a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ - local min;

(b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ - local max;

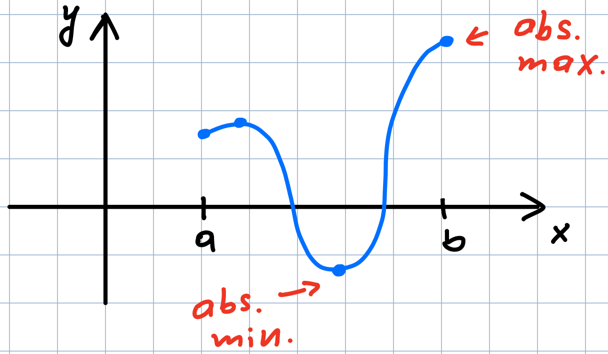
(c) If $D<0$, then $f(a,b)$ is a saddle point of f.

Rmk: If $D=0$, the test gives no information.

Rmk: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$

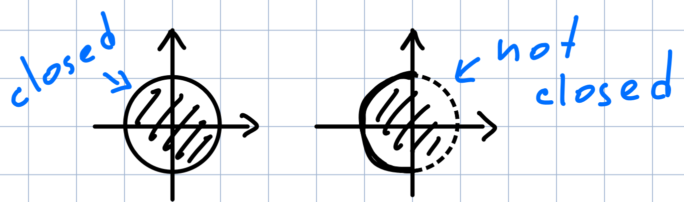
Extrema on bounded regions

Recall: $f(x)$ continuous on $[a,b]$, has an absolute max. and an abs. min. on $[a,b]$.



- We find them by evaluating f on critical points AND on endpoints.

$\mathcal{D}$ - closed set in $\mathbb{R}^2$ (i.e. contains boundary pts); bounded set (contained in some disk)



f, continuous on a closed, bounded $\mathcal{D} \subset \mathbb{R}^2$, attains an abs. max. value $f(x_1, y_1)$ and an abs. min. value $f(x_2, y_2)$ at some points in $\mathcal{D}$.

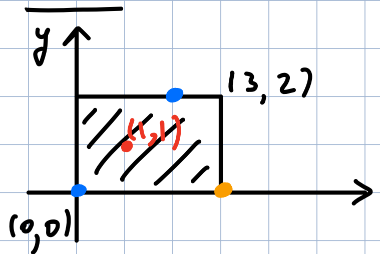
To find abs. max/min values of f on a closed, bounded $\mathcal{D}$:

1. Find values of f at critical points in $\mathcal{D}$;
2. Find extreme values on the boundary of $\mathcal{D}$;
3. largest of 1, 2 – abs. max.
   smallest of 1, 2 – abs. min.

Ex: $f(x,y) = x^2 - 2xy + 2y$, Find abs. min & max values on $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$

rectangle

Sol:



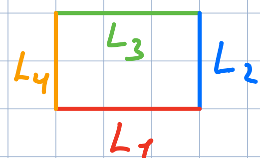
1) $f_x = 2x - 2y$ $\quad f_y = -2x + 2$

$$\begin{cases} 2x - 2y = 0 \\ -2x + 2 = 0 \end{cases} \Rightarrow \begin{cases} y = x \\ x = 1 \end{cases}$$

The only critical point is $(1,1)$

$f(1,1) = 1^2 - 2\cdot1\cdot1 + 2\cdot1 = 1$ - critical value

2) Boundary: 4 line segments



On L_1: $f(x,0) = x^2$ increasing

$0 \le x \le 3$ $\quad$ max: $f(3,0) = 9$; min: $f(0,0) = 0$

On L_2: $f(3,y) = 9 - 6y + 2y = \underbrace{9 - 4y}_{\text{decreasing}}$

$0 \le y \le 2$ $\quad$ min: $f(3,2) = 1$; max: $f(3,0) = 9$

On L_3: $f(x,2) = x^2 - 4x + 4 = (x-2)^2 \ge 0$ $\quad$ min: $f(2,2) = 0$;

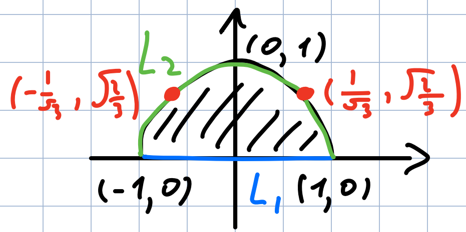
$0 \le x \le 3$ $\quad$ max: $f(0,2) = 4$.

On L_4: $f(0,y) = 2y$ increasing $\quad$ max: $f(0,2) = 4$

$0 \le y \le 2$ $\quad$ min: $f(0,0) = 0$

$$\boxed{\text{Abs. max: } f(3,0) = 9; \quad \text{Abs. min: } f(0,0) = f(2,2) = 0}$$

Ex: $f(x,y) = xy^2$ on $\mathcal{D} = \left\{ (x,y) \,\middle|\, \begin{matrix} x^2+y^2 \le 1 \\ y \ge 0 \end{matrix} \right\}$ find abs. max. & min.



Sol: 1) $f_x = y^2$ $\qquad \begin{cases} y^2 = 0 \\ 2xy = 0 \end{cases} \qquad \begin{cases} y = 0 \\ x \text{ any} \end{cases}$

$f_y = 2xy$

No critical points in the interior of $\mathcal{D}$.

$f(x,0) = 0$

2) On L_1: $y = 0$; $-1 \le x \le 1$ $\qquad f(x,0) = 0$

On L_2: $x^2+y^2 = 1$; $y \ge 0$ $\qquad \Rightarrow y = \sqrt{1-x^2}$

$\Rightarrow f = x(1-x^2) \qquad -1 \le x \le 1$

$g(x) = x - x^3, \quad -1 \le x \le 1$

$\frac{d}{dx} g(x) = 1 - 3x^2 \qquad g'(x) = 0$ iff $\underbrace{x = \pm \frac{1}{\sqrt{3}}}_{\text{critical pts. of } g}$

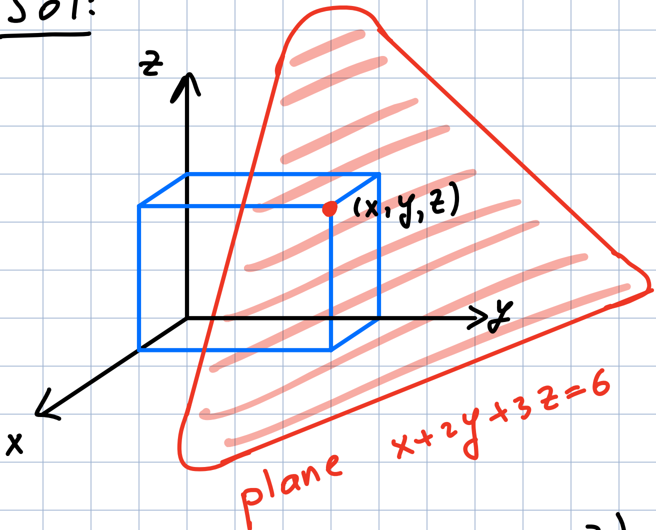
$g\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}$

$g\left(-\frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} = -\frac{2}{3\sqrt{3}}$

$\boxed{\text{Abs. max.: } f\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) = \frac{2}{3\sqrt{3}}; \ \text{Abs. min.: } f\left(-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) = -\frac{2}{3\sqrt{3}}}$

Ex: Find the volume of the largest rectangular box in the 1^{st} octant, with faces parallel to xy, yz, xz-planes and one vertex in the plane $x+2y+3z=6$.

Sol:

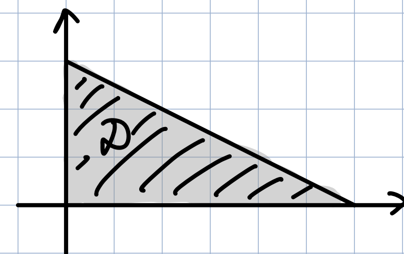


1) $z = \frac{6-x-2y}{3}$ $\qquad$ Volume $= x\cdot y\cdot z$

$$\text{Volume} = f(x,y) = xy\,\frac{6-x-2y}{3}$$

We want to maximize f in

$$\mathcal{D} = \left\{ (x,y) : \begin{matrix} x+2y \le 6 \\ x\ge 0,\ y\ge 0 \end{matrix} \right\}$$



2) $f_x(x,y) = y\,\frac{6-x-2y}{3} - \frac{1}{3}xy$

$f_y(x,y) = x\,\frac{6-x-2y}{3} - \frac{2}{3}xy$

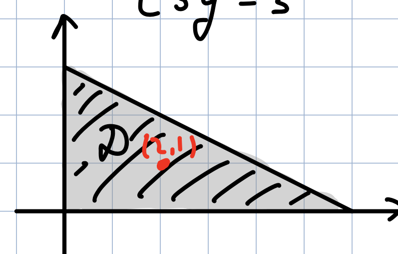
$$\begin{cases} y\,\frac{6-2x-2y}{3} = 0 \\ x\,\frac{6-x-4y}{3} = 0 \end{cases}$$

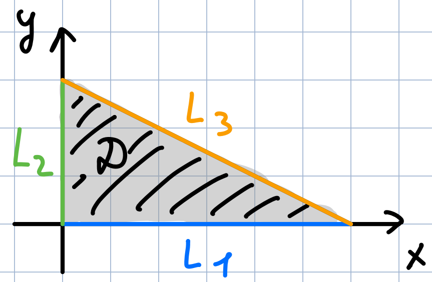
$x=0$ or $y=0$ $\leftarrow$ on the boundary of $\mathcal{D}$

$$\begin{cases} 6-2x-2y = 0 \\ 6-x-4y = 0 \end{cases} \iff \begin{cases} x+y=3 \\ x+4y=6 \end{cases} \qquad \begin{cases} x = 3-y \\ 3y = 3 \end{cases}$$

$(2,1)$ - critical point in the interior of $\mathcal{D}$

$$f(2,1) = 2\cdot 1\cdot \frac{6-2-2\cdot 1}{3} = \frac{4}{3}$$





On L_1: $F(x,0)=0$, $0 \le x \le 6$

On L_2: $F(0,y)=0$, $0 \le y \le 3$

On L_3: $0 \le y \le 3$ $\quad x = 6-2y$

$x+2y=6$

$$F(x,y) = xy\,\frac{6-x-2y}{3} = (6-2y)\,y\,\frac{\overbrace{6-(6-2y)-2y}^{=0}}{3} = 0$$

$\boxed{\text{Abs. max.: } F(2,1)=\frac{4}{3}, \text{ box of size } \underset{x}{2}\times\underset{y}{1}\times\underset{z}{\frac{2}{3}}}$